Equating the variation of the functional $I_1(v_1)$ to zero, we obtain the equation and initial condition for determining $v_1(t, x)$

$$\rho_0 v_{1tt} - \bar{a} v_{1xx} = (2c)^{-1} (\rho_0 \varepsilon_\rho v_{0ttt} + a_0 \varepsilon_a v_{0xxt}) \tag{11}$$
$$v_{1t}(0, x) = \tfrac{1}{2} c (\varepsilon_\rho + \varepsilon_a) f_{xx}(x)$$

to which we should add the initial condition $v_1(0, x) = 0$ following from the constraint (6). In addition to producing the function $v_0(t, x)$ (10), Eq.(11) yields the solution of the problem in question. We note that the approach adopted here does not give rise to ill-posed problems.

Relations (10) and (11) can be combined within the limits of accuracy used, into a single equation in terms of the function $v(x, t)$ sought

$$\rho_0 v_{tt} - \bar{a} v_{xx} - (2c)^{-1} a_0 (\varepsilon_\rho + \varepsilon_a) v_{xxt} = 0, \quad v(0, x) = f(x),$$
$$v_t(0, x) = g(x) + \tfrac{1}{2} c (\varepsilon_a + \varepsilon_\rho) f_{xx}(x)$$

It has the form of an equation of motion of a one-dimensional viscoelastic medium. Its solution, with the above initial conditions, yields an asymptotically exact value for the averaged solution of the initial equation (1) when $t \gg c^{-1}(\varepsilon_a + \varepsilon_\rho)$.

When the values of time t are nearly zero, the averaged solution has been shown to have the character of a boundary layer, and more complicated equations are needed for its determination, obtained by varying the functional (8). This explains the appearance of the last term in the second initial condition, which is not present in the exact formulation by virtue of relation (2) and of the definition of the averaged solution. The term in question describes the effect of the temporary boundary layer on the behaviour of the solution at finite times.

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THE BUBNOV-GALERKIN METHOD IN THE NON-LINEAR THEORY OF HOLLOW, FLEXIBLE MULTILAYER ORTHOTROPIC SHELLS*

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The existence of solutions of a strongly non-linear system of differential equations describing, in the framework of the kinematic Timoshenko model /1/ adopted for the whole packet in toto /2/, the behaviour of a flexible, multilayer shell whose very layer is made of an inhomogeneous orthotropic material, is proved. To obtain an approximate solution of the problem in question, a procedure is proposed and justified, using the Bubnov-Galerkin (BG) method based on constructing an auxiliary quasilinear system of equations. A similar approach makes it possible to extend the method /3–6/ of studying the convergence of the BG method to strongly non-linear systems of elliptic type equations, and to achieve the convergence of the sequence of approximate solutions to the exact solution in a space of any prescribed degree of smoothness, without imposing additional constraints on the initial data of the problem.

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We formulate the initial problem as follows. It is required to find, in the region $\Omega \subset E_2$ (E_2 is an Euclidean space and (x, y) is a point in E_2) with boundary $\partial\Omega$, satisfying the coditions which guarantee the application of the Sobolev inclusion theorems /7/, a solution of the system of differential equations with boundary conditions

$$
\begin{aligned}
&L_1(u) \equiv -\frac{\partial}{\partial x}(T_1) - \frac{\partial}{\partial y}(S) - P_x = 0 \qquad (1)\\
&L_2(v) \equiv -\frac{\partial}{\partial y}(T_2) - \frac{\partial}{\partial x}(S) - P_y = 0\\
&L_3(w) \equiv -k_x T_1 - k_y T_2 - \frac{\partial}{\partial x}(Q_1) - \frac{\partial}{\partial y}(Q_2) - \frac{\partial}{\partial x}\left(T_1 \frac{\partial w}{\partial x}\right) -\\
&- \frac{\partial}{\partial y}\left(T_2 \frac{\partial w}{\partial y}\right) - \frac{1}{2}\frac{\partial}{\partial y}\left(S \frac{\partial w}{\partial x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(S\frac{\partial w}{\partial y}\right) - q = 0\\
&L_4(\gamma_x) \equiv -\frac{\partial}{\partial x}(M_{11}) - \frac{\partial}{\partial y}(M_{12}) + Q_1 = 0\\
&L_5(\gamma_y) \equiv -\frac{\partial}{\partial y}(M_{22}) - \frac{\partial}{\partial x}(M_{12}) + Q_2 = 0\\
&u = v = w = \gamma_x = \gamma_y = 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

where

$$
\begin{aligned}
&T_\lambda = C_{1\lambda}\varepsilon_{11} + C_{\lambda 2}\varepsilon_{22} + K_{1\lambda}\varkappa_{11} + K_{\lambda 2}\varkappa_{22}, \quad S = C_{66}\varepsilon_{12} + K_{66}\varkappa_{12}\\
&M_{\lambda 1} = K_{1\lambda}\varepsilon_{11} + K_{\lambda 2}\varepsilon_{22} + D_{1\lambda}\varkappa_{11} + D_{\lambda 2}\varkappa_{22}, \quad M_{12} = K_{66}\varepsilon_{12} + D_{66}\varkappa_{12}\\
&Q_\lambda = A_{\lambda\lambda}\varepsilon_{\lambda 3}; \quad \lambda = 1, 2\\
&\varepsilon_{11} = \frac{\partial u}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{22} = \frac{\partial v}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\
&\varepsilon_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \quad \varepsilon_{13} = \gamma_x + \frac{\partial w}{\partial x}, \quad \varepsilon_{23} = \gamma_y + \frac{\partial w}{\partial y}\\
&\varkappa_{11} = \frac{\partial \gamma_x}{\partial x}, \quad \varkappa_{22} = \frac{\partial \gamma_y}{\partial y}, \quad \varkappa_{12} = \frac{\partial \gamma_x}{\partial y} + \frac{\partial \gamma_y}{\partial x}
\end{aligned}
$$

and $A_{\lambda\lambda}(x, y)$ are functions of rigidity, defined as follows:

in the case of an odd number of layers of constant thickness symmetrically distributed about the middle surface $z = 0$ /2/ we have

$$
A_{\lambda\lambda}(x, y) = G_{\lambda 3}^{m-1} \int_{-h_{m-1}}^{h_{m-1}} f(z)\,dz + 2\sum_{s=1}^{m} G_{\lambda 3}^{s} \int_{h_{s+1}}^{h_s} f(z)\,dz \qquad (2)
$$

in the case of an arbitrary number of layers of constant thickness /2/ we have

$$
A_{\lambda\lambda}(x, y) = \sum_{s=1}^{m+n} G_{\lambda 3}^{s} \int_{\delta_{s-1}-\Delta}^{\delta_s - \Delta} f(z)\,dz \qquad (3)
$$

and in the case of layers of variable thickness /2/ we have (3) with $\delta_s = \delta_s(x, y)$, $\Delta = 0$; $\lambda = 1, 2$.

We use the following notation: $u(x, y), v(x, y), w(x, y)$ are the displacements of the point of the middle surface along the lines x, y, z, respectively, $\gamma_x(x, y), \gamma_y(x, y)$ are the angles of rotation of the normal in the planes xz, yz, respectively, $k_x(x, y), k_y(x, y)$ are the curvatures of the middle plane, $P_x(x, y)$, $P_y(x, y)$ are the longitudinal load intensities, $q(x, y)$ is the transverse load intensity, T_1, T_2, S are the tangential forces, M_{11}, M_{22}, M_{12} are the bending and torsional moments, Q_1, Q_2 are the transverse forces, $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ are the tensile and shear deformations of the middle surface, $\varepsilon_{13}, \varepsilon_{23}$ are the transverse shear deformations, $\varkappa_{11}, \varkappa_{22}, \varkappa_{12}$ are the bending deformations, $G_{ij}(x, y), K_{ij}(x, y), D_{ij}(x, y)$ are known functions of the rigidity, /2/, $G_{13}(x, y), G_{23}(x, y)$ are the shear moduli in the xz, yz planes respectively independent of the variable z, $f(z)$ is the distribution function of tangential stresses over the shell thickness and /1/, h_i, δ_i, Δ are constants in the formulas (2), (3) characterizing the thickness and position of each layer in the shell /2/. The functions $C_{ij}, K_{ij}, D_{ij}, A_{ij}$ satisfy the conditions

$$
0 < \alpha_1 \leqslant C_{ij} \leqslant \beta_1, \; 0 < \alpha_2 \leqslant A_{ij} \leqslant \beta_2, \; 0 < \alpha_3 \leqslant K_{ij} \leqslant \beta_3, \; 0 < \alpha_4 \leqslant D_{ij} \leqslant \beta_4 \qquad (4)
$$

by definition.

We shall use the following notation for the Sobolev spaces:

$$
\begin{aligned}
&W_2^m(\Omega) = \{u \mid D^\alpha u \in L_2(\Omega), \; \forall \alpha \mid \alpha \mid \leqslant m\}, \; W_2^{\circ 1}(\Omega) =\\
&\{u \mid u \in W_2^1(\Omega), \; u = 0 \quad \text{on} \quad \partial\Omega\}, \; H_1 = [W_2^{\circ 1}]^5\\
&H_2 = [W_2^{\circ 1}]^2 \times W_2^2 \cap W_2^{\circ 1} \times [W_2^{\circ 1}]^2
\end{aligned}
$$

$(\cdot, \cdot)$ is a scalar product in $W_2^{\circ}(\Omega) \equiv L_2(\Omega)$, $|\cdot|_M$ is the norm in the Hilbert space M, in particular $|\cdot|_{01}$ is the norm in $W_2^{\circ 1}(\Omega)$, $|\cdot|_m$ is the norm in $W_2^m(\Omega)$. We introduce in H_1 and H_2 as follows:

$$
\begin{aligned}
&|\cdot|_{H_1}^2 = |\cdot|_{01}^2 + |\cdot|_{01}^2 + |\cdot|_{01}^2 + |\cdot|_{01}^2 + |\cdot|_{01}^2,\\
&|\cdot|_{H_2}^2 = |\cdot|_{01}^2 + |\cdot|_{01}^2 + |\cdot|_{2}^2 + |\cdot|_{01}^2 + |\cdot|_{01}^2
\end{aligned}
$$

We shall call the vector $\omega = (u, v, w, \gamma_x, \gamma_y) \in H_1$ satisfying the integral identities

$$(L_1(u), \varphi_1) \equiv \left(T_1, \frac{\partial \varphi_1}{\partial x}\right) + \left(S, \frac{\partial \varphi_1}{\partial y}\right) - (P_x, \varphi_1) = 0 \tag{5}$$
$$(L_2(v), \varphi_2) \equiv \left(T_2, \frac{\partial \varphi_2}{\partial y}\right) + \left(S, \frac{\partial \varphi_2}{\partial x}\right) - (P_y, \varphi_2) = 0$$
$$(L_3(w), \varphi_3) \equiv (T_1, -k_x\varphi_3) + (T_2, -k_y\varphi_3) + \left(A_{11}\left(\gamma_x + \frac{\partial w}{\partial x}\right), \frac{\partial \varphi_3}{\partial x}\right) +$$
$$\left(A_{22}\left(\gamma_y + \frac{\partial w}{\partial y}\right), \frac{\partial \varphi_3}{\partial y}\right) + \left(T_1, \frac{\partial w}{\partial x}\frac{\partial \varphi_3}{\partial x}\right) + \left(T_2, \frac{\partial w}{\partial y}\frac{\partial \varphi_3}{\partial y}\right) +$$
$$\frac{1}{2}\left(S, \frac{\partial w}{\partial x}\frac{\partial \varphi_3}{\partial y}\right) + \frac{1}{2}\left(S, \frac{\partial w}{\partial y}\frac{\partial \varphi_3}{\partial x}\right) - (q, \varphi_3) = 0$$
$$(L_4(\gamma_x), \varphi_4) \equiv \left(M_{11}, \frac{\partial \varphi_4}{\partial x}\right) + \left(M_{12}, \frac{\partial \varphi_4}{\partial y}\right) + \left(A_{11}\left(\gamma_x + \frac{\partial w}{\partial x}\right), \varphi_4\right) = 0$$
$$(L_5(\gamma_y), \varphi_5) \equiv \left(M_{22}, \frac{\partial \varphi_5}{\partial y}\right) + \left(M_{12}, \frac{\partial \varphi_5}{\partial x}\right) + \left(A_{22}\left(\gamma_y + \frac{\partial w}{\partial y}\right), \varphi_5\right) = 0,$$
$$\forall \varphi \in H_1, \quad \varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5)$$

the generalized solution of problem (1).

We shall consider, together with (1), the following auxiliary problem:

$$L_1(u) = P_x,\ L_2(v) = P_y,\ L_3(w) + \varepsilon\Delta^2 w = q,\ L_4(\gamma_x) = 0,\ L_5(\gamma_y) = 0 \tag{6}$$
$$u = v = w = \gamma_x = \gamma_y = 0, \quad \Delta w - \frac{1-\nu}{\rho}\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

where $\partial/\partial n$ is a derviative along the upper normal, ρ is the radius of curvature of the contour $\partial\Omega$, ν is a positive constant and Δ is the Laplace operator.

We shall call the generalized solution of problem (6) the vector $\omega_\varepsilon = (u_\varepsilon, v_\varepsilon, w_\varepsilon, \gamma_{x\varepsilon}, \gamma_{y\varepsilon})$ satisfying the integral identity

$$(L_1(u_\varepsilon), \psi_1) + (L_2(v_\varepsilon), \psi_2) + (L_3(w_\varepsilon), \psi_3) + (L_4(\gamma_{x\varepsilon}), \psi_4) + \tag{7}$$
$$(L_5(\gamma_{y\varepsilon}), \psi_5) + \varepsilon\iint_\Omega \left(\Delta w_\varepsilon \Delta\psi_3 + 2(1-\nu)\left(\frac{\partial^2 w_\varepsilon}{\partial x\,\partial y}\frac{\partial^2\psi_3}{\partial x\,\partial y} - \right.\right.$$
$$\left.\left.\frac{1}{2}\frac{\partial^2 w_\varepsilon}{\partial x^2}\frac{\partial^2\psi_3}{\partial y^2} - \frac{1}{2}\frac{\partial^2 w_\varepsilon}{\partial y^2}\frac{\partial^2\psi_3}{\partial x^2}\right)\right) d\Omega = \iint_\Omega (P_x\psi_1 + P_y\psi_2 + q\chi_3)\, d\Omega,$$
$$\forall\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in H_2$$

Theorem 1. Let

$$k_x(x,y),\ k_y(x,y),\ P_x(x,y),\ P_y(x,y),\ q(x,y),\ A_{ij}(x,y), \tag{8}$$
$$C_{ij}(x,y),\ D_{ij}(x,y),\ K_{ij}(x,y) \in L_2(\Omega)$$
$$\alpha_1 - \beta_3 > 0,\ \alpha_4 - \beta_3 > 0,\ \alpha_1 2 - 2\beta_3 > 0,\ \alpha_4 2 - 2\beta_3 > 0$$

Then: 1) for any $\varepsilon > 0$ there exists at least one vector $\omega_\varepsilon^\circ = (u_\varepsilon^\circ, v_\varepsilon^\circ, w_\varepsilon^\circ \gamma_{x\varepsilon}^\circ, \gamma_{y\varepsilon}^\circ)$ satisfying the identity (7); 2) the approximate solution of problem (6) can be found with the aid of the BG method in the form (the summation over repeated indices is carried out from 1 to n)

$$u_\varepsilon^n = \sum a_i\chi_i,\quad v_\varepsilon^n = \sum b_j\chi_j,\quad w_\varepsilon^n = \sum c_k\chi_{1k},\quad \gamma_{x\varepsilon}^n = \sum d_l\chi_l,\quad \gamma_{y\varepsilon}^n = \sum e_p\chi_p \tag{9}$$

where $\{\chi_{1_i}(x,y)\}$, $\{\chi_i(x,y)\}$ are the basis systems in $W_2^2(\Omega) \cap W_2^{\circ 1}(\Omega)$ and $W_2^{\circ 1}(\Omega)$, respectively. Also $v_\varepsilon^n \to u_\varepsilon^\circ$, $v_\varepsilon^n \to v_\varepsilon^\circ$, $\gamma_{x\varepsilon}^n \to \gamma_{x\varepsilon}^\circ$, $\gamma_{y\varepsilon}^n \to \gamma_{y\varepsilon}^\circ$ weakly in $W_2^{\circ 1}(\Omega)$ $w_\varepsilon^n \to w_\varepsilon$ weakly in $W_2^2(\Omega) \cap W_2^{\circ 1}(\Omega)$ and strongly in $W_2^{\circ 1}(\Omega)$.

Proof. We will obtain the approximate solution of problem (6) using the BG procedure, determining the coefficients in (9) from the following system of equations:

$$(L_1(u_\varepsilon^n), \chi_i) + (L_2(v_\varepsilon^n), \chi_j) + (L_3(w_\varepsilon^n), \chi_{1k}) + (L_4(\gamma_{x\varepsilon}^n), \chi_l) + \tag{10}$$
$$(L_5(\gamma_{y\varepsilon}^n), \chi_p) + \varepsilon\iint_\Omega \Delta w_\varepsilon^n \mu\chi_{1k} + 2(1-\nu)\left(\frac{\partial^2 w_\varepsilon^n}{\partial x\,\partial y}\frac{\partial^2\chi_{1k}}{\partial x\,\partial y} - \right.$$
$$\left.\left.\frac{1}{2}\frac{\partial^2 w_\varepsilon^n}{\partial x^2}\frac{\partial^2 w_\varepsilon^n}{\partial y^2} - \frac{1}{2}\frac{\partial^2 w_\varepsilon^n}{\partial y^2}\frac{\partial^2\chi_{1j}}{\partial x^2}\right)\right) d\Omega = \iint_\Omega (P_x\chi_i + P_y\chi_j +$$
$$q\chi_{1k})\, d\Omega, \quad i, j, k, l, p = 1, 2, \ldots, n$$

The solvability of the system follows from the lemma "on the acute angle" /8/. Indeed, let us introduce, as in /8/, the mapping $P(C) \equiv (L_1(C), L_2(C), L_3(C), L_4(C), L_5(C))\colon R \to R$ where $R = [C^n]^5$, C^n is a Banach space of continuous functions of n variables. The continuity of the mapping $L_j(C)$ $(j = 1, 2, \ldots, 5)$ is obvious (the continuity of the non-linear terms follows from the compactness of the inclusion $W_4^1(\Omega)$ into $W_2^2(\Omega)$. We shall show that the "acute angle" condition holds. To do this we multiply every equation of system (10) by the corresponding factor

a_i, b_j, c_k, d_l, e_p and sum over i, j, k, l, p from 1 to n

$$(P(C), C) \geqslant |\sqrt{C_{11}}\,\varepsilon_{11}^n|^2 + |\sqrt{C_{22}}\,\varepsilon_{22}^n|^2 + |\sqrt{C_{66}}\,\varepsilon_{12}^n|^2 + |\sqrt{D_{11}}\,\varkappa_{11}^n|^2 + |\sqrt{D_{22}}\,\varkappa_{22}^n|^2 + |\sqrt{D_{66}}\,\varkappa_{12}^n|^2 + \left|\sqrt{A_{11}}\left(\gamma_{x\varepsilon}^n + \frac{\partial w_\varepsilon^n}{\partial x}\right)\right|^2 + \left|\sqrt{A_{22}}\left(\gamma_{y\varepsilon}^n + \frac{\partial w_\varepsilon^n}{\partial y}\right)\right|^2 + 2(C_{12}\varepsilon_{22}^n, \varepsilon_{11}^n) + 2(D_{12}\varkappa_{22}^n, \varkappa_{11}^n) + 2(K_{11}\varkappa_{11}^n, \varepsilon_{11}^n) + 2(K_{12}\varkappa_{22}^n, \varepsilon_{11}^n) + 2(K_{12}\varkappa_{11}^n, \varepsilon_{22}^n) + 2(K_{22}\varkappa_{22}^n, \varepsilon_{22}^n) + 2(K_{66}\varkappa_{12}^n, \varepsilon_{12}^n) + c\,|\sqrt{\varepsilon}\,w_\varepsilon^n|^2_{W_2^1(\Omega)} - ((P_x, u_\varepsilon^n) + (P_y, v_\varepsilon^n) + (q, w_\varepsilon^n)), \quad c = \text{const} > 0 \tag{11}$$

where $\varepsilon_{11}^n, \varepsilon_{22}^n, \varepsilon_{12}^n, \varkappa_{11}^n, \varkappa_{22}^n, \varkappa_{12}^n$ are obtained from $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varkappa_{11}, \varkappa_{22}, \varkappa_{12}$ by replacing the vector $\omega = (u, v, w, \gamma_x, \gamma_y)$ by the vector $\omega_\varepsilon^n = (u_\varepsilon^n, v_\varepsilon^n, w_\varepsilon^n, \gamma_{x\varepsilon}^n, \gamma_{y\varepsilon}^n)$.

The definition of the coefficients C_{ij}, D_{ij} /2/ and condition (6) together imply that $C_{\lambda\lambda} - C_{12} \geqslant \alpha_1/2$, $D_{\lambda\lambda} - D_{12} \geqslant \alpha_4/2$ $(\lambda = 1, 2)$. Using the Cauchy inequality with «ε» /7/ and the Cauchy-Bunyakovskii inequality /7/ for the last term in (11), we have

$$(P(C), C) \geqslant \left(\frac{\alpha_1}{2} - 2\beta_3\right)(|\varepsilon_{11}^n|^2 + |\varepsilon_{22}^n|^2) + (\alpha_1 - \beta_3)|\varepsilon_{12}^n|^2 + \left(\frac{\alpha_4}{2} - 2\beta_3\right)(|\varkappa_{11}^n|^2 + |\varkappa_{22}^n|^2) + (\alpha_4 - \beta_3)|\varkappa_{12}^n|^2 + \alpha_2\left(\left|\gamma_{x\varepsilon}^n + \frac{\partial w_\varepsilon^n}{\partial x}\right|^2 + \left|\gamma_{y\varepsilon}^n + \frac{\partial w_\varepsilon^n}{\partial y}\right|^2\right) + c\,|\sqrt{\varepsilon}\,w_\varepsilon^n|^2_{W_2^1(\Omega)} - c_1(|P_x||u_\varepsilon^n| + |P_y||v_\varepsilon^n| + |q||w_\varepsilon^n|) \tag{12}$$

Thus when $|C|$ is sufficiently large and the condition of Theorem 1 is taken into account, the acute angle condition holds $(P(C), C) > 0$. This allows us to assert that system (10) has a solution, and enables us to write the following a priori estimates for the set of approximate solutions:

$$|u_\varepsilon^n|_{01} \leqslant c^0, \quad |v_\varepsilon^n|_{01} \leqslant c^0, \quad |\sqrt{\varepsilon}\,w_\varepsilon^n|_2 \leqslant c^0$$
$$|\gamma_{x\varepsilon}^n|_{01} \leqslant c^0, \quad |\gamma_{y\varepsilon}^n|_{01} \leqslant c^0, \quad c^0 = \text{const} > 0 \tag{13}$$

Using the estimates (13) and the theorem on the compactness of the inclusion $W_2^1(\Omega)$ into $W_2^2(\Omega)$, we carry out in the well-known manner /4, 5/ the passage to the limit from n in (10), and this completes the proof of the theorem.

The theorem which follows shows in what sense the solution of (6) approximates the solution of (1).

Theorem 2. Let the conditions of Theorem 1 hold. Then, as $\varepsilon \to 0$, a subsequence $\{u_\varepsilon, v_\varepsilon, w_\varepsilon, \gamma_{x\varepsilon}, \gamma_{y\varepsilon}\}$ can be found which converges to the solution $\omega^0 = (u^0, v^0, w^0, \gamma_x^0, \gamma_y^0)$ of problem (1) in the following sense: $u_\varepsilon \to u^0, v_\varepsilon \to v^0, \gamma_{x\varepsilon} \to \gamma_x^0, \gamma_{y\varepsilon} \to \gamma_y^0$ weakly in $\overset{\circ}{W}{}_2^1(\Omega)$, $w_\varepsilon \to w^0$ weakly in $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and $w_\varepsilon \to w^0$ strongly in $\overset{\circ}{W}{}_2^1(\Omega)$.

Proof. We note that we can obtain from (13) estimates for the approximate solutions of problem (6) by passing to the limit in n. The presence of these estimates enables us to choose a subsequence $\{\omega_\varepsilon\} = \{u_\varepsilon, v_\varepsilon, w_\varepsilon, \gamma_{x\varepsilon}, \gamma_{y\varepsilon}\}$ weakly convergent in H_2. This justifies, together with the Sobolev inclusion theorems, the possibility of a passage to the limit as $\varepsilon \to 0$ in the integral identity (5) after the preliminary closure of the set $\{\chi_i\} \equiv H_2$ on the norm of H_1.

Notes. 1°. Condition (8) will always hold in Theorem 1 for the layers of constant thickness, whether they are symmetrically or arbitrarily distributed, provided that the coordinate surface /2/ is chosen appropriately, and in the case of layers of variable thickness condition (8) holds, in particular, when the layers are distributed symmetrically about the coordinate surface $z = 0$.

2°. We can consider, as the auxiliary problem (6), the problem in which the biharmonic operator is replaced by an arbitrary, positive definite operator T with natural boundary conditions, whose energy space is imbedded compactly in $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. We note that the freedom in the choice of the operator T enables us "to construct", in a known sense, the properties of the resulting algebraic system in the BG method in order to increase the computational efficiency of the algorithm used.

The introduction of the auxiliary problem (6) makes is possible to obtain strong convergence of some sequence of approximate solutions w_ε^n to the exact solution w^0, which is important from the point of view of the practical realization of the algorithm. A similar result can be obtained also for the other functions sought u, v, γ_x, γ_y, provided that we complement the expressions $L_1(u), L_2(v), L_4(\gamma_x), L_5(\gamma_y)$ in systems (6) with terms of the form $\varepsilon_1 T_1 u$, $\varepsilon_2 T_2 v$, $\varepsilon_3 T_3 \gamma_x$, $\varepsilon_4 T_4 \gamma_y$, respectively, where $\varepsilon_i > 0$ $(i = 1, 2, 3, 4)$, T_i are positive definite operators with natural boundary conditions whose energy space is imbedded compactly in $W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$. Clearly, by choosing the operators T, T_i appropriately, we can obtain practically any degree of convergence of the subsequence of approximate solutions to the exact solution without imposing any additional constraints on the initial data of problem (1).

3°. A proof analogous to the one given above holds for other boundary vlaues (e.g. when the character of the boundary condition varies along the contour /9/), naturally, when the boundary conditions for the Timoshenko-type model are transferred appropriately.

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